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# Duality and Lower Bounds for Relative Projection Constants

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This paper discusses the use of a duality theorem for the computation of lower bounds for relative projection constants. A first application yields a slight improvement of a lower bound which has already been proved by the author. Further applications concern polynomial projection in the $L_1$-space as well as polynomial projections in the multivariate case. The derived lower bounds are asymptotically best possible. © 1995 Academic Press, Inc.

## 1. INTRODUCTION

A very simple but nevertheless a very powerful tool for estimating values of linear operators is the Lebesgue inequality. Let $L$ be a linear mapping from a normed linear space $X$ into a second one, say $Y$, and let the subspace $U$ of $X$ be in the kernel of $L$; then the Lebesgue inequality reads as

$$\|L[f]\| \leqslant \|L\| \cdot \operatorname{dist}(f, U),$$

where

$$\operatorname{dist}(f, U) = \inf_{u \in U} \|f-u\|.$$

In the approximation theory, we mostly have the situation that $X$ is a space of functions and $Y = U$ is a subspace of $X$, such that $L$ is the error of a projection $P$ from $X$ onto $Y$. Under rather general conditions, we have

$$\|L\| = 1 + \|P\|$$

(cf. Cheney and Price [3, Theorem 9]). Hence, projections with small norms play a very important role. Although, in several settings, much effort

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has been made to find a minimal projection, i.e., a projection with a minimal norm among all projections from $X$ onto $Y$, only a few important minimal projections are known explicitly. Even in a case such as $X = C[-1, 1]$ (endowed with the supremum norm) and $U = Y = \mathbb{P}_2$ (the space of all polynomials of degree at most 2), which, at first glance, appears to be very simple, it took a long time to determine the minimal projection and that projection has a complicated structure (cf. Chalmers and Metcalf [1]). It therefore seems to be hopeless to find minimal projections for settings such as $X = C[-1, 1]$ and $U = Y = \mathbb{P}_n$ for arbitrary $n$.

A somewhat weaker problem is the determination of projections with *small norms*. For this purpose, we have to say what a small norm is. Let $\mathscr{P}(X, Y)$ denote the set of all projections from $X$ onto $Y$ and let $\varrho(Y, X)$ be the corresponding relative projection constant, i.e., the infimum over all norms of projections in $\mathscr{P}(X, Y)$. We define the quality of a projection $P_0$ relative to the norm $\|\cdot\|$ by the equation

$$\mathrm{qual}(P_0, \|\cdot\|, X, Y) = \frac{\|P_0\|}{\varrho(Y, X)}.$$

The smaller the quality of $P_0$, the better $P_0$ is relative to the norm. If the quality equals 1, $P_0$ is a minimal projection, and if the quality is near 1, we should not spend too much effort to find the minimal projection, if it seems to be too complicated. For a sequence $(Y_n)_{n \in \mathbb{N}}$ of spaces, we call a sequence $(P_n)_{n \in \mathbb{N}}$ of projections asymptotically minimal, if the sequence of the numbers $\mathrm{qual}(P_n, \|\cdot\|, X, Y_n)$ converges to 1.

The calculation of relative projection constants in general appears to be as hopeless as the determination of a minimal projection. Nevertheless, to estimate the quality from above, we at least need good lower bounds for relative projection constants. Our aim is to determine such lower bounds in a rather systematic way. The basic idea is described in Section 2, while in Section 3 applications to several spaces of functions and several subspaces of polynomials in one or more dimensions are given.

## 2. The Method for Obtaining Lower Bounds for Projection Constants

First, let us suppose for the following that the subspace $U$ is complemented in the respective space $X$, i.e., that there exists at least one projection from $X$ onto $U$. We then start with the observation that the set of all projections from $X$ onto the subspace $U$ is an affine subspace of the space $\mathscr{L}(X, U)$ of all continuous linear operators from $X$ into $U$. Thus, we fix an arbitrary projection $P_0$ such that each projection $P$ has the (unique) representation $P = P_0 - A$, where $A$ is an element of the closed linear subspace $\mathscr{A}(X, U)$ of $\mathscr{L}(X, U)$,

$$\begin{aligned}\mathscr{A}(X, U) &:= \{P_1 - P_2 \mid P_\nu \in \mathscr{P}(X, U)\} \\ &= \{A \in \mathscr{L}(X, U) \mid A[u] = 0 \text{ for all } u \in U\}.\end{aligned}$$

We have

$$\begin{aligned}\varrho(U, X) &= \inf_{P \in \mathscr{P}(X, U)} \|P\| \\ &= \inf_{P \in \mathscr{P}(X, U)} \|P_0 - (P_0 - P)\| \\ &= \inf_{A \in \mathscr{A}(X, U)} \|P_0 - A\| \\ &= \operatorname{dist}(P_0, \mathscr{A}(X, U)).\end{aligned}$$

We are now in the typical approximation theoretical situation that we want to estimate the error of a best approximation from below. The standard method for doing this is the following application of the Hahn–Banach theorem (see Singer [9, Theorem 1.1]).

*Let $\mathscr{Z}(X, U)$ be the set of all elements of the unit ball of the dual space $\mathscr{L}(X, U)^*$, which annihilate the operators in $\mathscr{A}(X, U)$; then*

$$\varrho(U, X) = \sup_{H \in \mathscr{Z}(X, U)} |H[P_0]|, \tag{1.1}$$

*where $P_0$ is an arbitrary projection from $X$ onto $U$.*

Hence, each functional in $\mathscr{Z}(X, U)$ yields a lower bound for the relative projection constant and we just have to find an appropriate functional $H$.

The equation (1.1) has been used (e.g., in Light and Cheney [7]) for characterizing minimal projections, but, in the following, it also proves to be succesful in obtaining lower bounds.

## 3. Applications to Polynomial Projections

We will consider some applications of the principle presented in Section 2.

The first one concerns the relative projection constants $\varrho(\mathbb{P}_n, C[-1, 1])$, where $C[-1, 1]$ is endowed with the supremum norm. In [8], rather the same lower bounds for these projection constants were proved. The proof below is similar, but the version here may elucidate more clearly the theoretical background. The lower bounds are asymptotically sharp in the sense that they allow the determination of asymptotically minimal projections, which are, e.g., the projections $C_n^{(1)}$ onto the partial sums of the

Chebyshev expansion of the first kind. Note that the norm of $C_n^{(1)}$ is equal to the Lebesgue constant $L_n$ being the norm of the $n$th Fourier partial sum operator.

THEOREM 1 (See also [8]). *Let $C_n^{(1)}$ be the nth partial sum operator of the Chebyshev expansion of the first kind; then,*

$$\frac{4}{\pi^2}(\ln n - \ln\ln n) + \frac{1}{3} \leqslant \varrho(\mathbb{P}_n, C[-1,1]) \leqslant \|C_n^{(1)}\| \leqslant \frac{4}{\pi^2}\ln(2n+1) + 1.$$

In the second example, we investigate the relative projection constants $\varrho(\mathbb{P}_n, [-1,1])$. The best-known lower bounds were proved by Görlich and Markett [6]. They can be improved by factors of approximately 2. Again, our new bounds will be asymptotically best possible. The projections $C_n^{(2)}$ onto partial sums of Chebyshev expansions of the second kind are asymptotically minimal.

THEOREM 2. *Let $C_n^{(2)}$ be the nth partial sum operator of the Chebyshev expansion of the second kind; then,*

$$\frac{4}{\pi^2}(\ln n - \ln\ln n) + \frac{1}{4} \leqslant \varrho(\mathbb{P}_n, L_1[-1,1]) \leqslant \|C_n^{(2)}\| = L_{n+1} \leqslant \frac{4}{\pi^2}\ln(2n+3) + 1.$$

Our third example is a simple but nontrivial generalization of Theorem 1 to the multivariate case. Let $\mathbb{P}_{\mathbf{n}} = \bigotimes_{i=1}^{m} \mathbb{P}_{n_i} \subset C([-1,1]^m)$ be the space of polynomials, which are of degree less than or equal to $n_i$ in the $i$th variable. Furthermore, let $C_{\mathbf{n}}$ be the generalized Chebyshev projection from $C([-1,1]^m)$ onto $\mathbb{P}_{\mathbf{n}}$,

$$C_{\mathbf{n}}[f](\mathbf{x}) = \left(\frac{2}{\pi}\right)^m \int_{[-1,1]^m} \sum_{k_i \leqslant n_i} \prod_{i=1}^{m} T_{k_i}^*(x_i)\, T_{k_i}^*(t_i)(1-t_i)^{-1/2}\, d\mathbf{t},$$

where $T_k^* = T_k$ if $k \geqslant 1$ and $T_0^*(x) \equiv \sqrt{1/2}$.

THEOREM 3.

$$\prod_{i=1}^{m}\left(\frac{4}{\pi^2}(\ln n_i - \ln\ln n_i) + \frac{1}{3}\right) \leqslant \varrho(\mathbb{P}_{\mathbf{n}}, C([-1,1]^m)) \leqslant \|C_{\mathbf{n}}\| \leqslant \prod_{i=1}^{m}\left(\frac{4}{\pi^2}\ln(2n_i+1) + 1\right),$$

*i.e., the generalized Chebyshev projections are asymptotically minimal if each of the $n_i$ is increasing.*

The proofs of all the theorems are based on a certain property of the corresponding Dirichlet kernel. Let $I \subset \mathbb{Z}^m$ be a finite set indices. Then, the (generalized) Dirichlet kernel is defined by

$$D_I(\mathbf{t}) = 2^m \sum_{(\nu_1, ..., \nu_m) \in I} \prod_{i=1}^{m} \cos^* \nu_i t_i,$$

where the asterisk means that the cosine has to be halved if $\nu_i = 0$. The corresponding set of polynomials shall be denoted by $\mathbb{P}_I$ and it contains all linear combinations of the terms $T_{\nu_1}(x_1) \cdots T_{\nu_m}(x_m)$, where $(\nu_1, ..., \nu_m) \in I$. The definition of the corresponding Chebyshev projection, $C_I$, is as follows.

$$C_I[f](\cos x_1, ..., \cos x_m)$$
$$= \frac{1}{(2\pi)^m} \int_{[0, \pi]^m} f(\cos t_1, ..., \cos t_m) \sum D_I(t_1 \pm x_1, ..., t_m \pm x_m) \, d\mathbf{t},$$

where the summation runs over all combinations of signs in the argument of $D_I$.

DEFINITION 1. We call a sequence of functions $f_n \colon \mathbb{R}^m \to \mathbb{R}$ local, if, for arbitrary $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{\int_{[-\varepsilon, \varepsilon]^m} |f_n(\mathbf{x})| \, d\mathbf{x}}{\int_{[-\pi, \pi]^m} |f_n(\mathbf{x})| \, d\mathbf{x}} = 1.$$

LEMMA 1. *Let $I(n)$ be index sets such that the $D_{I(n)}$ form a local sequence of Dirichlet kernels; then, the corresponding generalized Chebyshev projections are asymptotically minimal in $\mathscr{P}(C([-1, 1]^m), \mathbb{P}_{I(n)})$.*

Let $I(n, m) = \{(\nu_1, ..., \nu_m) \mid \nu_i \geqslant 0, \sum_{i=1}^{m} \nu_i \leqslant n\}$. Then, we denote by $C_n^m$ the Chebyshev projection with image $\mathbb{P}_n^m := \mathbb{P}_{I(n, m)}$. The image is thus the space of polynomials of maximal degree $n$ in $m$ variables. No simple expression is known for the Dirichlet kernel, and therefore for the corresponding kernel function of the Chebyshev projection, if $m > 2$. For $m = 1$ we have the usual Dirichlet kernel and for $m = 2$ an explicit expression was given by Daugavet [4]. Nevertheless, we can prove that the Dirichlet kernels $D_n^m$ are local.

THEOREM 4. *The Chebyshev projections $C_n^m$ are asymptotically minimal in $\mathscr{P}(C[-1, 1]^m), \mathbb{P}_n^m)$.*

*Remark.* Theorems 3 and 4, as well as Lemma 1, also hold for projections from the corresponding $L_1$-spaces onto polynomials. (In the lower bound in Theorem 3, the constant 1/3 then has to be replaced by 1/4).

## 4. Proof of the Results

*Proof of Theorem* 1. The last inequality in Theorem 1 is known (see Watson [10]). To prove the lower estimate, we note that there is a one-to-one norm-preserving mapping from $\mathscr{P}(C[-1, 1], \mathbb{P}_n)$ to $\mathscr{P}(C^e[0, \pi], \mathbb{T}_n^e)$, where $C^e[0, \pi]$ denotes the space of all even, continuous, $2\pi$-periodic functions and where

$$\mathbb{T}_n^e = \left\{ t \mid t(x) = \sum_{\nu=0}^{n} a_\nu \cos \nu x, a_\nu \in \mathbb{R} \right\}$$

(cf. Cheney [2, p. 214]). However, the notation in $C^e[0, \pi]$ is slightly simpler and we will therefore prove the theorem in the latter setting.

First define the even $2\pi$-periodic function $h$ by

$$h(t) = h_n(t) = \begin{cases} \operatorname{sgn} D_n(t) & \text{if } t \in [0, \varepsilon] \\ 0 & \text{if } t \in ]\varepsilon, \pi], \end{cases} \qquad 0 < \varepsilon < \pi/2, \tag{4.1}$$

and where $D_n$ is the Dirichlet kernel,

$$D_n(x) = 1 + 2 \sum_{\nu=1}^{n} \cos \nu x = \frac{\sin[(n + 1/2) x]}{\sin(x/2)}.$$

For a given $\delta > 0$, we obtain a continuous, even, $2\pi$-periodic function $h^*$ by replacing the discontinuities of $h$ by "steep" line segments in such a way that $h^*$ agrees on $[0, \pi]$ with $h$ except on intervals of total length $\delta$. If the steep line segments have their zeros in the discontinuities of $h$ and if $\delta$ is sufficiently small, we see that

$$\int_0^\pi |h(t) - h^*(t)| \, dt = \delta/2.$$

Define

$$g^*(x, y) = g_x^*(y) = h^*(x + y) + h^*(x - y).$$

Since $h^*$ is a Lipschitz function, it has a uniformly convergent Fourier series, $h^* = \sum_{r=0}^{\infty} a_r m_r$, where $m_r(t) = \cos rt$, such that

$$g^*(x, y) = \sum_{r=0}^{\infty} a_r[\cos r(x + y) + \cos r(x - y)] = 2 \sum_{r=0}^{\infty} a_r \cos rx \cos ry,$$

the series being uniformly convergent for all $x$ and $y$. For any bounded operator $P$ from $C^e[0, \pi]$ to itself, now define

$$H[P] = \int_0^\pi P[g_x^*](x)\,dx.$$

We verify two points:

(1) If $A$ is in $\mathscr{A}(C^e[0, \pi], \mathbb{T}_n^e)$, then $H[A] = 0$.

Such an $A$ can be written as a finite sum of rank-one operators mapping onto span $m_j$ for $0 \leqslant j \leqslant n$. So consider $A$ defined by

$$A[m_r] = \begin{cases} 0 & \text{for} \quad 0 \leqslant r \leqslant n \\ b_r m_j & \text{for} \quad r > n. \end{cases}$$

Then

$$A[g_x^*](x) = 2 \sum_{r=n+1}^{\infty} a_r b_r \cos rx \cos jx,$$

the series being uniformly convergent for all $x$. The orthogonality of $m_r$, and $m_j$ now gives $H[A] = 0$.

(2) $\|H\| \leqslant \int_0^\pi \|g_x^*\|\,dx \leqslant \pi + 2(\varepsilon + \delta)$.

Recall $h^*(t) = 0$ for $\varepsilon + \delta < t < \pi$. If $\varepsilon + \delta < x < \pi - \varepsilon - \delta$, then $h(x + y) = 0$ for $0 \leqslant y \leqslant \pi$, hence $\|g_x^*\|_\infty = h(x - x) = 1$. For other $x$, we certainly have $\|g_x^*\|_\infty \leqslant 2$, so the statement follows.

Denote by $S_n^e$ the even part of the Fourier partial sum operator, i.e.,

$$S_n^e[f](x) = \frac{1}{\pi} \int_0^\pi f(y)[D_n(x + y) + D_n(x - y)]\,dt.$$

Using $\|D_n\|_\infty = 2n + 1$, $\int_a^{a+2\pi} D_n(t) = 2\pi$, and $\int_0^\varepsilon \operatorname{sgn} D_n(t)\,dt \geqslant 0$, we now obtain

$$\begin{aligned} H[S_n^e] &= \frac{1}{2\pi} \int_0^\pi \int_0^\pi \{h^*(x + y) + h^*(x - y)\} \\ &\qquad \times \{D_n(x + y) + D_n(x - y)\}\,dy\,dx \\ &= \frac{1}{2\pi} \int_0^\pi \int_x^{x+\pi} h^*(t)\{D_n(t) + D_n(2x - t)\}\,dt\,dx \\ &\quad + \frac{1}{2\pi} \int_0^\pi \int_{x-\pi}^{x} h^*(t)\{D_n(t) + D_n(2x - t)\}\,dt\,dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \int_0^\pi \int_{-\varepsilon}^{\varepsilon} h^*(t)\, D_n(t)\, dt\, dx \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} h^*(t) \int_0^\pi D_n(2x-t)\, dx\, dt \\
&\geqslant \frac{1}{2\pi} \int_0^\pi \int_{-\varepsilon}^{\varepsilon} h(t)\, D_n(t)\, dt\, dx \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} h(t) \int_0^\pi D_n(2x-t)\, dx\, dt - (n+1)\,\delta \\
&= \pi L_n - \int_\varepsilon^\pi |D_n(t)|\, dt + \tfrac{1}{2} \int_0^{2\pi} h(t)\, dt - (n+1)\,\delta \\
&\geqslant \pi L_n - \int_\varepsilon^\pi |D_n(t)|\, dt - (n+1)\,\delta.
\end{aligned}
$$

For the third and the fourth equality, we used that both the functions $h$ and $D_n$ are even and $2\pi$-periodic. We use the estimate from [8],

$$
\int_\varepsilon^\pi |D_n(t)|\, dt \leqslant \frac{4}{2n+1} \csc \frac{\varepsilon}{2} + \frac{4}{\pi} \ln \frac{4}{\varepsilon},
$$

where we choose $\varepsilon = \pi/2 \ln n$, and note that $\delta$ may be chosen arbitrarily close to 0. This yields the lower bound in Theorem 1. ∎

*Proof of Theorem* 2. The last two relations in Theorem 2 are known (see, e.g., Görlich and Markett [6] and Watson [10]).

We take the function $h = h_{n+1}$ from the proof of Theorem 1 and modify it again to obtain the continuous function $h^*$, which has the uniformly convergent Fourier series $\sum_{r=0}^{\infty} a_r m_r$. Defining

$$
g^*(\cos x, \cos y) = \tilde{g}(x, y) = \frac{1}{\pi + 2\varepsilon} (h^*(y-x) - h^*(y+x)),
$$

we obtain

$$
g^*(x, y) = 2 \sum_{r=0}^{\infty} a_r\, U_{r-1}(x)\, U_{r-1}(y) \sqrt{(1-x^2)(1-y^2)},
$$

where $U_\nu$ is the $\nu$th Chebyshev polynomial of the second kind. For any projection $P \in \mathscr{P}(L_1[-1, 1], \mathbb{P}_n)$, we define the functional $H$ by

$$
H[P] = \int_{-1}^{1} P\left[\frac{g^*(x, \cdot)}{\sqrt{1-\cdot^2}}\right](x)\, dx.
$$

Since $P$ may be written as

$$P[f](x)=\int_{-1}^{1} p(x, y) f(y)\, dy,$$

the norm of $P$ is given by

$$\|P\|=\sup_{y} \int_{-1}^{1} |p(x, y)|\, dx$$

(cf. Franchetti and Cheney [5]), from which we may conclude that

$$\begin{aligned}\|H\| &=\int_{0}^{\pi} \sup_{x \in[-1,1]}\left|\frac{g(x, y)}{\sqrt{1-y^{2}}}\right| dy \\ &=\int_{0}^{\pi} \sup_{x \in[0, \pi]} |\tilde{g}(x, y)|\, dy \leqslant \pi+2(\varepsilon+\delta).\end{aligned}$$

Each operator $A \in \mathscr{A}(L_1[-1,1], \mathbb{P}_n)$ can written as a finite sum of rank-one operators mapping onto span $U_j$ for $0 \leqslant j \leqslant n$. Consider therefore $A$ defined by

$$A[U_r]=\begin{cases}0 & \text{for } \ 0 \leqslant r \leqslant n \\ b_r U_j & \text{for } \ r>n.\end{cases}$$

Then

$$A\left[\frac{g^{*}(x, \cdot)}{\sqrt{1-\cdot^{2}}}\right](x)=2 \sum_{r=0}^{\infty} a_r b_{r-1} U_{r-1}(x)\, U_j(x) \sqrt{1-x^{2}}.$$

The orthogonality of the Chebyshev polynomials now yields $H[A]=0$. Using $\int_0^{\varepsilon} \operatorname{sgn} D_{n+1}(t)\, dt \leqslant 4\pi/(2n+3)$, we now obtain

$$\begin{aligned} H[C_n^{(2)}]= & \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{\pi}\{h^{*}(y-x)-h^{*}(y+x)\} \\ & \times\{D_{n+1}(y-x)-D_{n+1}(y+x)\}\, dy\, dx \\ = & \frac{1}{2\pi} \int_{0}^{\pi} \int_{x-\pi}^{x} h^{*}(t)\{D_{n+1}(t)-D_{n+1}(2x-t)\}\, dt\, dx \\ & +\frac{1}{2\pi} \int_{0}^{\pi} \int_{x}^{x+\pi} h^{*}(t)\{D_{n+1}(t)-D_{n+1}(2x-t)\}\, dt\, dx \\ = & \frac{1}{2\pi} \int_{0}^{\pi} \int_{-\varepsilon}^{\varepsilon} h^{*}(t)\, D_{n+1}(t)\, dt\, dx \\ & -\frac{1}{2\pi} \int_{0}^{2\pi} h^{*}(t) \int_{0}^{\pi} D_{n+1}(2x-t)\, dx\, dt\end{aligned}$$

$$\geqslant \frac{1}{2\pi} \int_0^{\pi} \int_{-\varepsilon}^{\varepsilon} h(t)\, D_{n+1}(t)\, dt\, dx$$

$$- \frac{1}{2\pi} \int_0^{2\pi} h(t) \int_0^{\pi} D_{n+1}(2x - t)\, dx\, dt - (n+2)\,\delta$$

$$= \pi L_{n+1} - \int_{\varepsilon}^{\pi} |D_{n+1}(t)|\, dt - \frac{1}{2} \int_0^{\pi} h(t)\, dt - (n+2)\,\delta$$

$$\geqslant \pi L_{n+1} - \int_{\varepsilon}^{\pi} |D_{n+1}(t)|\, dt - \frac{2\pi}{2n+3} - (n+2)\,\delta.$$

As in the proof of Theorem 1, the result follows. ∎

*Remark*. For simplicity, we use the function $h$ instead of the continuous modified functions $h^*$ in the following proofs. The modifications are obvious.

We denote the Fourier partial sum corresponding to a certain index set $I$ by $S_I$. Let $\mathbf{x} = (x_1, ..., x_m)$, then we define the even Fourier partial sum by

$$S_I^e[f](\mathbf{x}) = 2^{-m} \sum S_I[f](\pm x_1, ..., \pm x_m),$$

where summation runs over all combinations of signs. For any projection $P^{\text{trig}}$ onto a space of even multivariate trigonometric polynomials, we may define a projection $P^{\text{pol}}$ into a related polynomial space by

$$P^{\text{pol}}[g](\cos x_1, ..., \cos x_m) = P^{\text{trig}}[f](x_1, ..., x_m),$$

where

$$g(t_1, ..., t_m) = f(\cos t_1, ..., \cos t_m).$$

Both operators have the same supremum norm. Therefore, it suffices to prove the results for projections onto spaces of even trigonometric polynomials. A multivariate Chebyshev partial sum operator is thus related to a Fourier partial sum operator and its Dirichlet kernel.

*Proof of Theorem* 3. The projection $S_{\mathbf{n}}^e$ has the representation

$$S_{\mathbf{n}}^e[f](x) = \frac{1}{(2\pi)^m} \int_{[0,\pi]^m} f(\xi_1, ..., \xi_m) \prod_{i=1}^{m} (D_{n_i}(\xi_i + x_i) + D_{n_i}(\xi_i - x_i))\, d\xi.$$

Let $\varepsilon_i$ depend on $n_i$ in the same way as $\varepsilon$ depends on $n$ in the proof of Theorem 1 and let $h_n$ be defined by (4.1). Then, we set

$$g(\mathbf{x}, \xi) = \prod_{i=1}^{m} \frac{1}{\pi + 2\varepsilon_i} (h_{n_i}(\xi_i + x_i) + h_{n_i}(\xi_i - x_i))$$

and

$$H[P]=\int_{[0,\pi]^m} P[g(\mathbf{x},\cdot)](\mathbf{x})\,d\mathbf{x}. \tag{4.2}$$

We have $H\in\mathscr{L}(C^e([0,\pi]^m),\mathbb{T}_{\mathbf{n}}^e)$ and obtain

$$H[S_{\mathbf{n}}^e]\cdot\prod_{i=1}^m(\pi+2\varepsilon_i)=\frac{1}{(2\pi)^m}\int_{[0,\pi]^{2m}}\prod_{i=1}^m(D_{n_i}(\xi_i+x_i)+D_{n_i}(\xi_i-x_i))$$
$$\times(h_{n_i}(\xi_i+x_i)+h_{n_i}(\xi_i-x_i))\,d\mathbf{x}\,d\xi.$$

Completely analogous to the proof of Theorem 1, this may be estimated below by

$$\prod_{i=1}^m\left(\pi\,\|C_{n_i}^{(1)}\|-\int_{\varepsilon_i}^{\pi}|D_{n_i}(t)|\,dt\right),$$

which yields the lower estimate of the theorem. ▮

*Proof of Lemma* 1. We set $I=I(n)$ and define

$$h_I(\mathbf{x})=\begin{cases}\operatorname{sgn} D_I(\mathbf{x}) & \text{if } \mathbf{x}\in[-\varepsilon,\varepsilon]^m\\ 0 & \text{if } \mathbf{x}\in[-\pi,\pi]^m\backslash[-\varepsilon,\varepsilon]^m\end{cases}$$

and

$$g(\mathbf{x},\xi)=(\pi+2e)^{-m}\sum h(\xi_1\pm x_1,\ldots,\xi_m\pm x_m),$$

where the summation runs over all combinations of signs between $\xi_i$ and $x_i$. Then, the functional $H$, defined as in (4.2), is in $\mathscr{L}(C^e([0,\pi]^m),\mathbb{T}_I^e)$. We obtain

$$(\pi+2\varepsilon)^m\,H[S_I^e]=\frac{1}{(2\pi)^m}\int_{[0,\pi]^{2m}}\Big(\sum D_I(\xi_1\pm x_1,\ldots,\xi_m\pm x_m)$$
$$\times h_I(\xi_1\pm x_1,\ldots,\xi_m\pm x_m)\Big)\,d\xi\,d\mathbf{x}.$$

We first consider the case that, in this expression, $D_I$ and $h_I$ have a different sign in a corresponding argument $\xi_i\pm x_i$. Without restriction, let $\xi_1+x_1$ appear as an argument of $D_I$ and let $\xi_1-x_1$ appear as an argument of $h_I$. Then,

$$\left|\int_{[0,\pi]^{2m}} \Big(D_I(\xi_1+x_1, \xi_2 \pm x_2, ..., \xi_m \pm x_m)\right.$$

$$\left.\times h_I(\xi_1 - x_1, \xi_2 \pm x_2, ..., \xi_m \pm x_m)\Big) \, d\xi \, d\mathbf{x}\right|$$

$$\leqslant \int_0^\pi \int_{\xi_1-\pi}^{\xi_1} \int_{[0,\pi]^{2m-2}} |D_I(2\xi_1 - t, \xi_2 \pm x_2, ..., \xi_m \pm x_m)$$

$$\times h_I(t, \xi_2 \pm x_2, ..., \xi_m \pm x_m)| \, d\xi_2 \, dx_2 \cdots d\xi_m \, dx_m \, dt \, d\xi_1$$

$$\leqslant \int_0^\pi \int_{-\pi}^{\pi} \int_{[0,\pi]^{2m-2}} |D_I(2\xi_1 - t, \xi_2 \pm x_2, ..., \xi_m \pm x_m)$$

$$\times h_I(t, \xi_2 \pm x_2, ..., \xi_m \pm x_m)| \, d\xi_2 \, dx_2 \cdots d\xi_m \, dx_m \, dx_m \, dt \, d\xi_1$$

$$\leqslant 2\varepsilon \int_{[-\pi,\pi]} \int_{[0,\pi]^{2m-2}} |D_I(\xi_1, \xi_2 \pm x_2, ..., \xi_m \pm x_m)|$$

$$\times d\xi_2 \, dx_2 \cdots d\xi_m \, dx_m \, d\xi_1$$

$$\leqslant 2\varepsilon\pi^{m-1} \int_{[-\pi,\pi]^m} |D_I(\xi)| \, d\xi = \frac{2\varepsilon}{\pi} (2\pi^2)^m \, \|S_I^e\|.$$

We now consider the integral

$$\int_{[0,\pi]^{2m}} \Big(\sum D_I(\xi_1 \pm x_1, ..., \xi_m \pm x_m) \cdot h_I(\xi_1 \pm x_1, ..., \xi_m \pm x_m)\Big) \, d\xi \, d\mathbf{x}, \tag{4.3}$$

where the summation runs over all signs, where the arguments of $h_I$ and $D_I$ coincide. Then, for a fixed vector $\xi$, the integration of $D_I h_I$ with respect to $\mathbf{x}$ runs over an $m$-cube with edge length $2\pi$ and center in $\xi$. Since $D_I$ is $2\pi$-periodic in each variable, the integral in (4.3) equals

$$\pi^m \int_{[-\varepsilon,\varepsilon]^m} |D_I(\mathbf{x})| \, d\mathbf{x}.$$

The assumption that the $D_I = D_{I(n)}$ are local now yields

$$H[S_{I(n)}^e] \geqslant \|S_{I(n)}^e\| - 2^m\varepsilon \, \|S_{I(n)}^e\| + o(\|S_{I(n)}^e\|).$$

Since $\varepsilon > 0$ may be chosen arbitrarily, the lemma follows. ∎

*Proof of Theorem* 4. We have to prove that the Dirichlet kernels $D_n^m$ are local. It can be shown that

$$2^{m+1} D_n^m(\mathbf{x}) = \frac{\sin Nx_1}{\sin(1/2)\,x_1} \sum \mathrm{co}_n^{(m-1)}(x_1 \pm x_2, x_1 \pm x_3, ..., x_1 \pm x_m)$$
$$- \frac{\cos Nx_1}{\sin(1/2)\,x_1} \sum \mathrm{si}_n^{(m-1)}(x_1 \pm x_2, x_1 \pm x_3, ..., x_1 \pm x_m), \quad (4.4)$$

where the summation runs over all combinations of signs, $N = n + 1/2$, and where

$$\mathrm{co}_n^{(m)}(t_1, ..., t_m) = \sum_{\nu_1 + \cdots + \nu_m \leq n} \cos^*(\nu_1 t_1 + \cdots + \nu_m t_m)$$

and

$$\mathrm{si}_n^{(m)}(t_1, ..., t_m) = \sum_{\nu_1 + \cdots + \nu_m \leq n} \sin^*(\nu_1 t_1 + \cdots + \nu_m t_m).$$

Here, the asterisk means that the term has to be halved for each index $\nu_i$, which appears in the argument and equals 0.

At first, we prove the following lemma.

LEMMA 2. *The functions* $\mathrm{si}_n^{(m)}$ *and* $\mathrm{co}_n^{(m)}$ are local.

*Proof.* We show the lemma by induction over the parameter $m$. For $m = 1$ it is well-known. The step from $m$ to $m + 1$ is as follows.

Let $\mathbf{x} = (x_1, ..., x_m)$, then we have the recurrence relations

$$\mathrm{co}_n^{(m+1)}(x_0, \mathbf{x}) = \frac{1}{2\sin(1/2)\,x_0} \Big( \sin Nx_0 \,\mathrm{co}_n^{(m)}(x_1 - x_0, ..., x_m - x_0)$$
$$+ \cos Nx_0 \,\mathrm{si}_n^{(m)}(x_1 - x_0, ..., x_m - x_0) - \cos \frac{x_0}{2} \mathrm{si}_n^{(m)}(\mathbf{x}) \Big)$$

and

$$\mathrm{si}_n^{(m+1)}(x_0, \mathbf{x}) = \frac{1}{2\sin(1/2)\,x_0} \Big( \sin Nx_0 \,\mathrm{si}_n^{(m)}(x_1 - x_0, ..., x_m - x_0)$$
$$- \cos Nx_0 \,\mathrm{co}_n^{(m)}(x_1 - x_0, ..., x_m - x_0) + \cos \frac{x_0}{2} \mathrm{co}_n^{(m)}(\mathbf{x}) \Big).$$

Let $\varepsilon > 0$ be arbitrary and let $k \in \mathbb{N}$ and $\delta > 0$ be so chosen that $(k + 1)\,\delta < \varepsilon$. Then,

$$\begin{aligned}
s_\varepsilon &:= \int_{[-\varepsilon,\varepsilon]^{m+1}} |\mathrm{si}_n^{(m+1)}(x_0, \mathbf{x})| \, d\mathbf{x} \, dx_0 \\
&\geqslant \int_{2\delta}^{k\delta} \int_{[\delta,(k+1)\delta]^m} \frac{1}{2\sin(x_0/2)} \Big( \big|\, |\sin Nx_0 \, \mathrm{si}_n^{(m)}(x_1 - x_0, ..., x_m - x_0)| \\
&\quad - |\cos Nx_0 \, \mathrm{co}_n^{(m)}(x_1 - x_0, ..., x_m - x_0)| \,\big| - \left|\cos\frac{x_0}{2} \, \mathrm{co}_n^{(m)}(\mathbf{x})\right| \Big) \, d\mathbf{x} \, dx_0 \\
&\geqslant \int_{2\delta}^{k\delta} \frac{1}{2\sin(x_0/2)} \Big( \Big| |\sin Nx_0| \int_{[-\delta,\delta]^m} |\mathrm{si}_n^{(m)}(\mathbf{x})| \, d\mathbf{x} \\
&\quad - |\cos Nx_0| \int_{[-\delta,\delta]^m} |\mathrm{co}_n^{(m)}(\mathbf{x})| \, d\mathbf{x} \Big| \\
&\quad - \left|\cos\frac{x_0}{2}\right| \int_{[-\pi,\pi]^m \setminus [-\delta,\delta]^m} |\mathrm{co}_n^{(m)}(\mathbf{x})| \, d\mathbf{x} \Big) \, dx_0 \\
&\geqslant \int_{2\delta}^{k\delta} \frac{1}{2\sin(x_0/2)} \big|\, |\sin Nx_0| \, \|\mathrm{si}_n^{(m)}\|_1 - |\cos Nx_0| \, \|\mathrm{co}_n^{(m)}\|_1 \,\big| \, dx_0 \\
&\quad + o\left( \int_{2\delta}^{k\delta} \frac{1}{2\sin(x_0/2)} (\|\mathrm{si}_n^{(m)}\|_1 + \|\mathrm{co}_n^{(m)}\|_1) \, dx_0 \right).
\end{aligned}$$

Let $\alpha, \beta > 0$ and let

$$f(\alpha, \beta) := \int_0^\pi |\alpha \,|\sin x| - \beta \,|\cos x|\,| \, dx.$$

Then, using the characterization theorem for best $L_1$-approximations, we obtain

$$\begin{aligned}
f(\alpha, \beta) &\geqslant \max\{\min_t f(t, \beta), \min_t f(\alpha, t)\} \\
&= \max\left\{ f\left(\frac{\beta}{\sqrt{3}}, \beta\right), f\left(\alpha, \frac{\alpha}{\sqrt{3}}\right) \right\} \\
&= 2(\sqrt{3} - 1) \max\{\alpha, \beta\}.
\end{aligned}$$

This yields the estimate

$$\begin{aligned}
s_\varepsilon &\geqslant \frac{\sqrt{3}-1}{\pi} (1 + o(1)) \max\{\|\mathrm{si}_n^{(m)}\|_1, \|\mathrm{co}_n^{(m)}\|_1\} \int_{2\delta}^{k\delta} \csc\frac{x_0}{2} \, dx_0 \\
&\quad + o(\|\mathrm{si}_n^{(m)}\|_1 + \|\mathrm{co}_n^{(m)}\|_1) \int_{2\delta}^{k\delta} \csc\frac{x_0}{2} \, dx_0 \\
&\geqslant \frac{\sqrt{3}-1}{\pi} (1 + o(1)) (\|\mathrm{si}_n^{(m)}\|_1 + \|\mathrm{co}_n^{(m)}\|_1) \ln\frac{k}{2}.
\end{aligned}$$

On the other hand, we have

$$\left(\int_\varepsilon^\pi + \int_{-\pi}^{-\varepsilon}\right) \int_{[-\pi,\pi]^m} |\mathrm{si}_n^{(m+1)}(x_0, \mathbf{x})| \, d\mathbf{x} \, dx_0$$

$$\leqslant 2 \int_\varepsilon^\pi \frac{1}{2\sin(x_0/2)} (2\, \|\mathrm{co}_n^{(m)}\|_1 + \|\mathrm{si}_n^{(m)}\|_1) \, dx_0$$

$$\leqslant 4(\|\mathrm{co}_n^{(m)}\|_1 + \|\mathrm{si}_n^{(m)}\|_1) \ln \tan \frac{\varepsilon}{4}. \tag{4.5}$$

When replacing $x_0$ by any $x_i$, the same estimate also holds, such that

$$\int_{[-\pi,\pi]^m \setminus [-\varepsilon,\varepsilon]^m} |\mathrm{si}_n^{(m+1)}(x_0, \mathbf{x})| \, d\mathbf{x} \, dx_0$$

$$\leqslant 4(m+1)(\|\mathrm{co}_n^{(m)}\|_1 + \|\mathrm{si}_n^{(m)}\|_1) \ln \tan \frac{\varepsilon}{4}. \tag{4.6}$$

Since for each fixed $\varepsilon$ the number $k$ is arbitrary, we have that $\mathrm{si}_n^{(m+1)}$ is local. Analogously, it may be poved that $\mathrm{co}_n^{(m+1)}$ is also local. ∎

We are now in the position to prove that $D_n^m$ is local. As in the proof of Lemma 2, we choose $(k+1)\,\delta < \varepsilon$, such that

$$\int_{[-\varepsilon,\varepsilon]^m} |D_n^{(m)}(\mathbf{x})| \, d\mathbf{x} \geqslant \int_{2\delta}^{k\delta} \int_{[\delta,(k+1)\delta]^{m-1}} |D_n^{(m)}(\mathbf{x})| \, d\mathbf{x} =: d.$$

If, in the representation (4.4), a term $x_1 + x_i$ appears, then the argument of $\mathrm{si}_n^{(m-1)}$ resp. of $\mathrm{co}_n^{(m-1)}$ is not in $[-\delta, \delta]^{m-1}$, and hence,

$$\begin{aligned} d \geqslant \int_{2\delta}^{k\delta} \frac{1}{2^{m+1}\sin(x_1/2)} \Bigg( \Bigg| \int_{[\delta,(k+1)\delta]^{m-1}} & |\sin Nx_1 \\ & \times \mathrm{co}_n^{(m-1)}(x_1 - x_2, ..., x_1 - x_m)| \, dx_2 \cdots dx_m \\ & - \left| \int_{[\delta,(k+1)\delta]^{m-1}} \cos Nx_1 \, \mathrm{si}_n^{(m-1)}(x_1 - x_2, ..., x_1 - x_m) \right| dx_2 \cdots dx_m \Bigg| \\ & + o(\|\mathrm{co}_n^{(m-1)}\|_1 + \|\mathrm{si}_n^{(m-1)}\|_1) \Bigg) \, dx_1 \\ = \int_{2\delta}^{k\delta} \frac{1}{2^{m+1}\sin(x_1/2)} & (|\, |\sin Nx_1| \, \|\mathrm{co}_n^{(m-1)}\|_1 - |\cos Nx_1| \, \|\mathrm{si}_n^{(m-1)}\|_1 \,| \\ & + o(\|\mathrm{co}_n^{(m-1)}\|_1 + \|\mathrm{si}_n^{(m-1)}\|_1)) \, dx_1. \end{aligned}$$

This integral also appears in the proof of Lemma 2 and we obtain thus

$$d \geqslant \frac{\sqrt{3}-1}{2^m \pi}(1+o(1))(\|\mathrm{si}_n^{(m-1)}\|_1 + \|\mathrm{co}_n^{(m-1)}\|_1) \ln \frac{k}{2}.$$

The integral of $D_n^m$ over $[-\pi, \pi]^m \backslash [-\varepsilon, \varepsilon]^m$ is estimated as in the inequalities (4.5) and (4.6) above and the theorem is proved. ▮

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## REFERENCES

1. B. L. CHALMERS AND F. T. METCALF, Determination of a minimal projection from $C[-1, 1]$ onto the quadratics, *Number. Funct. Anal. Optim.* **11** (1990), 1–10.
2. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.
3. E. W. CHENEY AND K. H. PRICE, Minimal projections, *in* "Approximation Theory" (A. Talbot, Ed.), pp. 261–289, Academic Press, New York, 1970.
4. I. K. DAUGAVET, Lebesgue constants for double Fourier series, *Metody Vychisl.* **6** (1970), 8–13.
5. C. FRANCHETTI AND E. W. CHENEY, Minimal projections in $L_1$-spaces, *Duke Math. J.* **43** (1976), 501–510.
6. E. GÖRLICH AND C. MARKETT, A lower bound for projection operators on $L^1(-1, 1)$, *Ark. Mat.* **24** (1986), 81–92.
7. W. A. LIGHT AND E. W. CHENEY, The characterization of best approximations in tensor-product spaces, *Analysis* **4** (1984), 1–26.
8. K. PETRAS, On the minimal norms of polynomial projections, *J. Approx. Theory* **62** (1990), 206–212.
9. I. SINGER, "Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces," Springer-Verlag, Berlin, 1970.
10. G. A. WATSON, The constants of Landau and Lebesgue, *Quart. J. Math. Oxford Ser.* **1** (1930), 310–318.